

# Vierbein walls in condensed matter.

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The effective field, which plays the part of the vierbein in general relativity, can have topologically stable surfaces, vierbein domain walls, where the effective contravariant metric is degenerate. We consider vierbein walls separating domains with the flat space-time which are not causally connected at the classical level. Possibility of the quantum mechanical connection between the domains is discussed.

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## I. INTRODUCTION.

In some classes of superfluids and superconductors there is an effective field arising in the low-energy corner, which acts on quasiparticles as gravitational field. Here we discuss topological solitons in superfluids and superconductors, which represent the vierbein domain wall. At such surface in the 3D space (or at the 3D hypersurface in 3+1 space) the vierbein is degenerate, so that the determinant of the contravariant metric  $g^{\mu\nu}$  becomes zero on the surface. An example of the vierbein domain wall has been discussed in Ref. [1] for the  $^3\text{He-A}$  film. When such vierbein wall moves, it splits into a black hole/white hole pair, which experiences the quantum friction force due to Hawking radiation [1]. Here we discuss the stationary wall, which is topologically stable and thus does not experience any dissipation. Such domain walls, at which one of the three "speeds of light" crosses zero, can be realized in other condensed matter too: in superfluid  $^3\text{He-B}$  [2], in chiral  $p$ -wave superconductors [3,4], and in  $d$ -wave superconductors [5].

In the literature two types of the walls were considered: with degenerate  $g^{\mu\nu}$  and with degenerate  $g_{\mu\nu}$  [6]. The case of degenerate  $g_{\mu\nu}$  was discussed in details in [6,7]. Both types of the walls could be generic. According to Horowitz [8], for a dense set of coordinate transformations the generic situation is the 3D hypersurface where the covariant metric  $g_{\mu\nu}$  has rank 3.

The physical origin of the walls with the degenerate metric  $g^{\mu\nu}$  in general relativity have been discussed by Starobinsky [9]. They can arise after inflation, if the inflaton field has a  $Z_2$  degenerate vacuum. The domain walls separates the domains with 2 different vacua of the inflaton field. The metric  $g^{\mu\nu}$  can everywhere satisfy the Einstein equations in vacuum, but at the considered surfaces the metric  $g^{\mu\nu}$  cannot be diagonalized as  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . Instead, on such surface the metric is diagonalized as  $g^{\mu\nu} = \text{diag}(1, 0, -1, -1)$  and thus cannot be inverted. Though the space-time can be

flat everywhere, the coordinate transformation cannot remove such a surface: it can only move the surface to infinity. Thus the system of such vierbein domain walls divides the space-time into domains which cannot communicate with each other. Each domain is flat and infinite as viewed by a local observer living in a given domain. In principle, the domains can have different space-time topology, as is emphasized by Starobinsky [9].

Here we consider the vierbein walls separating the flat space-time domains, which classically cannot communicate with each other across the wall, and discuss the quantum mechanical behavior of the fermions in the presence of the domain wall.

## II. VIERBEIN DOMAIN WALL

The simplest example of the vierbein walls we are interested in is provided by the domain wall in superfluid  $^3\text{He-A}$  film which separates domains with opposite orientations of the unit vector  $\hat{\mathbf{l}}$  of the orbital momentum of Cooper pairs:  $\hat{\mathbf{l}} = \pm \hat{\mathbf{z}}$ . Here the  $\hat{\mathbf{z}}$  is along the normal to the film. The Bogoliubov-Nambu Hamiltonian for fermionic quasiparticles is

$$\mathcal{H} = \left( \frac{p_x^2 + p_y^2 - p_F^2}{2m} \right) \tau^3 + \mathbf{e}_1 \cdot \mathbf{p} \tau^1 + \mathbf{e}_2 \cdot \mathbf{p} \tau^2 \quad (1)$$

Here  $\tau^a$  are  $2 \times 2$  matrices for the Bogoliubov-Nambu spin;  $\mathbf{p} = \hat{\mathbf{x}}p_x + \hat{\mathbf{y}}p_y$  is the 2D momentum (for simplicity we assume that the film is narrow so that the motion along the normal to the film is quantized and only the motion along the film is free); the complex vector

$$\mathbf{e} = \mathbf{e}_1 + i\mathbf{e}_2, \quad \hat{\mathbf{l}} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} = \pm \hat{\mathbf{z}} \quad (2)$$

is the order parameter. If one considers the first term in Eq.(1) as the mass term, then the vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  play the part of the zweibein for the 2D motion in the film.

We assume the following order parameter texture in the wall:

$$\mathbf{e}_1(x) = \hat{\mathbf{x}}c_x(x) , \quad \mathbf{e}_2 = \hat{\mathbf{y}}c_y(x) . \quad (3)$$

where the "speed of light" propagating along the axis  $y$  is constant, while the "speed of light" propagating along the axis  $x$  changes sign across the wall:

$$c_y(x) = c_0 , \quad c_x(x) = c_0 \tanh \frac{x}{d} , \quad (4)$$

At  $x = 0$  the zweibein is degenerate: the vector product  $\mathbf{e}_1 \times \mathbf{e}_2 = 0$  so that the  $\hat{\mathbf{l}}$  vector is not determined.

Since the momentum projection  $p_y$  is the conserved quantity, we come to pure 1+1 motion. Further we assume that (i)  $p_y = \pm p_F$ ; and (ii) the parameters of the system are such that the thickness  $d$  of the domain wall is large:  $\frac{d \gg \hbar}{mc_0}$ . This allows us to consider the range of the momentum  $\hbar/d \ll p_x \ll mc_0$ , where the term  $p_x^2$  can be either neglected as compared to the linear term or considered in the semiclassical approximation. Then rotating the Bogoliubov spin and neglecting the noncommutativity of the  $p_x^2$  term and  $c(x)$  one has the following Hamiltonian for the 1+1 particle:

$$\mathcal{H} = M(\mathcal{P})\tau^3 + \frac{1}{2}(c(x)\mathcal{P} + \mathcal{P}c(x))\tau^1 , \quad (5)$$

$$M^2(\mathcal{P}) = \frac{\mathcal{P}^4}{4m^2} + c_0^2 p_y^2 . \quad (6)$$

where the momentum operator  $\mathcal{P}_x = -i\partial_x$  is introduced. If the  $\mathcal{P}^2$  term is completely neglected, one obtains the 1+1 Dirac fermions

$$\mathcal{H} = M\tau^3 + \frac{1}{2}(c(x)\mathcal{P} + \mathcal{P}c(x))\tau^1 , \quad (7)$$

$$M^2 = M^2(\mathcal{P} = 0) = c_0^2 p_y^2 . \quad (8)$$

The classical spectrum of quasiparticles,

$$E^2 - c^2(x)p_x^2 = M^2 , \quad (9)$$

corresponds to the contravariant metric

$$g^{00} = 1 , \quad g^{xx} = -c^2(x) . \quad (10)$$

The line element of the effective space-time is

$$ds^2 = dt^2 - (c(x))^{-2} dx^2 . \quad (11)$$

The metric element  $g_{xx}$  is infinite at  $x = 0$ .

The Eq.(11) represents a *flat* effective spacetime for any function  $c(x)$ . However the singularity at  $x = 0$ , where  $g_{xx} = \infty$ , cannot be removed by the coordinate transformation. If at  $x > 0$  one introduces a new coordinate  $\xi = \int dx/c(x)$ , then the line element takes the standard flat form

$$ds^2 = dt^2 - d\xi^2 . \quad (12)$$

However, the other domain – the half-space with  $x < 0$  – is completely removed by such transformation. The situation is thus the same as discussed by Starobinsky for the domain wall in the inflaton field [9].

The two flat spacetimes are disconnected in the relativistic approximation. However this approximation breaks down near  $x = 0$ , where the "Planck energy physics" becomes important and nonlinearity in the energy spectrum appears in Eq.(6): The two halves actually communicate due to the high-energy quasiparticles, which are superluminal and thus can propagate through the wall.

### III. FERMIONS ACROSS VIERBEIN WALL.

In classical limit the low-energy relativistic quasiparticles do not communicate across the vierbein wall, because the speed of light  $c(x)$  vanishes at  $x = 0$ . However the quantum mechanical connection can be possible. There are two ways to treat the problem. In one approach one makes the coordinate transformation first. Then in one of the domains, say, at  $x > 0$ , the line element is Eq.(12), and one comes to the standard solution for the Dirac particle propagating in flat space:

$$\chi(\xi) = \frac{A}{\sqrt{2}} \exp(i\xi\tilde{E}) \begin{pmatrix} Q \\ Q^{-1} \end{pmatrix} + \frac{B}{\sqrt{2}} \exp(-i\xi\tilde{E}) \begin{pmatrix} Q \\ -Q^{-1} \end{pmatrix} , \quad (13)$$

$$\tilde{E} = \sqrt{E^2 - M^2} , \quad Q = \left( \frac{E + M}{E - M} \right)^{1/4} . \quad (14)$$

Here  $A$  and  $B$  are arbitrary constants. In this approach it makes no sense to discuss any connection to the other domain, which simply does not exist in this representation.

In the second approach we do not make the coordinate transformation and work with both domains. The wave function for the Hamiltonian Eq.(8) at  $x > 0$  follows from the solution in Eq.(13) after restoring the old coordinates:

$$\chi(x > 0) = \frac{A}{\sqrt{2c(x)}} \exp(i\xi(x)\tilde{E}) \begin{pmatrix} Q \\ Q^{-1} \end{pmatrix} + \frac{B}{\sqrt{2c(x)}} \exp(-i\xi(x)\tilde{E}) \begin{pmatrix} Q \\ -Q^{-1} \end{pmatrix} , \quad (15)$$

$$\xi(x) = \int^x \frac{dx}{c(x)} \quad (16)$$

The similar solution exists at  $x < 0$ . We can now connect the solutions for the right and left half-spaces using (i) the analytic cotinuation across the point  $x = 0$ ; and (ii) the conservation of the quasiparticle current across the interface. The quasiparticle current e.g. at  $x > 0$  is

$$j = c(x)\chi^\dagger\tau^1\chi = |A|^2 - |B|^2. \quad (17)$$

The analytic cotinuation depends on the choice of the contour around the point  $x = 0$  in the complex  $x$  plane. Thus starting from Eq.(16) we obtain two possible solutions at  $x < 0$ . The first solution is obtained when the point  $x = 0$  is shifted to the lower half-plane:

$$\begin{aligned} \chi^I(x < 0) = & \\ & \frac{-iAe^{-\frac{\tilde{E}}{2T_H}}}{\sqrt{2|c(x)|}} \exp\left(i\xi(x)\tilde{E}\right) \begin{pmatrix} Q \\ Q^{-1} \end{pmatrix} + \\ & \frac{-iBe^{\frac{\tilde{E}}{2T_H}}}{\sqrt{2|c(x)|}} \exp\left(-i\xi(x)\tilde{E}\right) \begin{pmatrix} Q \\ -Q^{-1} \end{pmatrix}, \end{aligned} \quad (18)$$

where  $T_H$  is

$$T_H = \frac{\hbar}{2\pi} \left. \frac{dc}{dx} \right|_{x=0}. \quad (19)$$

The conservation of the quasiparticle current (17) across the point  $x = 0$  gives the connection between parameters  $A$  and  $B$ :

$$|A|^2 - |B|^2 = |B|^2 e^{\frac{\tilde{E}}{T_H}} - |A|^2 e^{-\frac{\tilde{E}}{T_H}}. \quad (20)$$

The quantity  $T_H$  looks like the Hawking radiation temperature determined at the singularity. As follows from Ref. [1] it is the limit of the Hawking temperature when the white hole and black hole horizons in the moving wall merge to form the static vierbein wall. Note, that there is no real radiation when the wall does not move. The parameter  $T_H/\tilde{E} \sim \partial\lambda/\partial x$ , where  $\lambda = 2\pi/p_x = (2\pi/\tilde{E})\partial c/\partial x$  is the de Broglie wavelength of the quasiparticle. Thus the quasiclassical approximation holds if  $T_H/\tilde{E} \ll 1$ .

The second solution is obtained when the point  $x = 0$  is shifted to the upper half-plane:

$$\begin{aligned} \chi^{II}(x < 0) = & \\ & \frac{iAe^{\frac{\tilde{E}}{2T_H}}}{\sqrt{2|c(x)|}} \exp\left(i\xi(x)\tilde{E}\right) \begin{pmatrix} Q \\ Q^{-1} \end{pmatrix} + \\ & \frac{iBe^{-\frac{\tilde{E}}{2T_H}}}{\sqrt{2|c(x)|}} \exp\left(-i\xi(x)\tilde{E}\right) \begin{pmatrix} Q \\ -Q^{-1} \end{pmatrix}, \end{aligned} \quad (21)$$

and the current conservation gives the following relation between parameters  $A$  and  $B$ :

$$|A|^2 - |B|^2 = |B|^2 e^{-\frac{\tilde{E}}{T_H}} - |A|^2 e^{\frac{\tilde{E}}{T_H}}. \quad (22)$$

Two solutions, the wave functions  $\chi^I$  and  $\chi^{II}$ , are connected by the relation

$$\chi^{II} \propto \tau_3(\chi^I)^* \quad (23)$$

which follows from the symmetry of the Hamiltonian

$$H^* = \tau_3 H \tau_3 \quad (24)$$

The general solution is the linear combination of  $\chi^I$  and  $\chi^{II}$

Though on the classical level the two worlds on both sides of the singularity are well separated, there is a quantum mechanical interaction between the worlds across the vierbein wall. The wave functions across the wall are connected by the relation  $\chi(-x) = \pm i\tau_3\chi^*(x)$  inspite of no possibility to communicate in the relativistic regime.

#### IV. NONLINEAR (NONRELATIVISTIC) CORRECTION.

In the above derivation we relied upon the analytic continuation and on the conservation of the quasiparticle current across the wall. Let us justify this using the nonlinear correction in Eq.(6), which was neglected before. We shall work in the quasiclassical approximation, which holds if  $\tilde{E} \gg T_H$ . In a purely classical limit one has the dispersion

$$E^2 = M^2 + c^2(x)p_x^2 + \frac{p_x^4}{4m^2}, \quad (25)$$

which determines two classical trajectories

$$p_x(x) = \pm \sqrt{2m \left( \sqrt{\tilde{E}^2 + m^2 c^4(x)} - mc^2(x) \right)}. \quad (26)$$

It is clear that there is no singularity at  $x = 0$ , the two trajectories continuously cross the domain wall in opposite directions, while the Bogoliubov spin continuously changes its direction. Far from the wall these two trajectories give the two solutions,  $\chi^I$  and  $\chi^{II}$ , in the quasiclassical limit  $\tilde{E} \gg T_H$ . The function  $\chi^I$

$$\chi^I(x > 0) = \frac{1}{\sqrt{2|c(x)|}} \exp\left(i\xi(x)\tilde{E}\right) \begin{pmatrix} Q \\ Q^{-1} \end{pmatrix}, \quad (27)$$

$$\chi^I(x < 0) = \frac{-i}{\sqrt{2|c(x)|}} \exp\left(-i\xi(x)\tilde{E}\right) \begin{pmatrix} Q \\ -Q^{-1} \end{pmatrix}. \quad (28)$$

describes the propagation of the quasiparticle from the left to the right without reflection at the wall: in the quasiclassical limit reflection is suppressed. The function  $\chi^{II}$  describes the propagation in the opposite direction:

$$\chi^{II}(x > 0) = \frac{1}{\sqrt{2|c(x)|}} \exp\left(-i\xi(x)\tilde{E}\right) \begin{pmatrix} Q \\ -Q^{-1} \end{pmatrix}, \quad (29)$$

$$\chi^{II}(x < 0) = \frac{i}{\sqrt{2|c(x)|}} \exp\left(i\xi(x)\tilde{E}\right) \begin{pmatrix} Q \\ Q^{-1} \end{pmatrix}, \quad (30)$$

The quasiparticle current far from the wall does obey the Eq.(17) and is conserved across the wall. This confirms the quantum mechanical connection between the spaces obtained in previous section.

In the limit of small mass  $M \rightarrow 0$ , the particles become chiral with the spin directed along or opposite to the momentum  $p_x$ . The spin structure of the wave function in a semiclassical approximation is given by

$$\chi(x) = e^{i\tau_2 \frac{\alpha}{2}} \chi(+\infty), \quad \tan \alpha = \frac{p_x}{2mc(x)}. \quad (31)$$

Since  $\alpha$  changes by  $\pi$  across the wall, the spin of the chiral quasiparticle rotates by  $\pi$ : the righthanded particle transforms to the lefthanded one when the wall is crossed.

## V. DISCUSSION.

It appears that there is a quantum mechanic coherence between the two flat worlds, which do not interact classically across the vierbein wall. The coherence is established by nonlinear correction to the spectrum of chiral particle:  $E^2(p) = c^2 p^2 + ap^4$ . The parameter  $a$  is positive in the condensed matter analogy, which allows the superluminal propagation across the wall at high momenta  $p$ . But the result does not depend on the magnitude of  $a$ : in the relativistic low energy limit the amplitudes of the wave function on the left and right sides of the wall remain equal in the quasiclassical approximation, though in the low energy corner the communication across the wall is classically forbidden. Thus the only relevant input of the "Planck energy" physics is the mere possibility of the superluminal communication between the worlds across the wall. That is why the coherence between particles propagating in two classically disconnected worlds can be obtained even in the relativistic domain, by using the analytic continuation and the conservation of the particle current across the vierbein wall.

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